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Prescribing Transient and Asymptotic Behaviour of LTI Systems with Stochastic Initial Conditions

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Abstract: This paper considers two different control problems for deterministic systems with stochastic initial conditions where, in addition to the usual asymptotic behavior requirement, we are interested in the transient behavior of the state distribution evolution. For the first one, we study control design such that the state trajectories enter target set at a given transient time with a prescribed minimum cumulative distribution. For the second one, we propose control design where the distribution of state variable at transient time is close to the target distribution. We illustrate the efficacy of the proposed solutions to the aforementioned control problems through numerical examples.

Keywords: control with prescribed transient behavior, control under stochastic initial condition, process control, nonlinear control design

1. INTRODUCTION

Deterministic dynamical models are widely used for control design. There are various forms of these models, where the continuous or discrete linear time-invariant (LTI) models are of the most common, simple and versatile. The primary interest for the control design tailored to these models is achieving asymptotic convergence to a desired state (e.g., the stability property), which is always possible in a global sense if the system is controllable. If this global asymptotic stability property holds, all initial values of the state will converge to the desired state as time goes to infinity. Accordingly, it is of more interest to consider the shaping of the transient behaviour for such a system, given a characteristic of the initial condition. In this paper, we will address systems that are subject to stochastic initial conditions, but are completely deterministic otherwise. Such systems are encountered often in practice, and two nice examples are robotic swarms (Cheah et al. (2009)) with a distribution on the initial position or smart manufacturing systems with variation in the initial material properties. The latter can include nano-scale manufacturing processes like ALD (Holmqvist et al. (2012)), where nano-scale variations in the wafer are significant. For background reading on Stochastic processes, we refer the reader to (Arnold (1990); Grimmett and Stirzaker (2001)). For the systems considered in this paper, the goal is to assign transient performance such that the variation stays within desired specification bounds.

In the literature, little attention has been given to transient performance for deterministic systems with stochastic initial conditions. Understandably, the majority of the attention has been given to transient and asymptotic performance for systems driven by Brownian motion and the associated field of stochastic control (Aström (1970), Bertsekas (1976)). Here, one of the main concerns is the shaping of the output probability density function (pdf) of a stochastic system (Sun (2006); Kárný (1996)), this is often referred to as stochastic distribution control. There are generally two approaches to this problem, where the first directly applies the Fokker-Planck or Kolmogorov forward equation (Risken (1984); Briat and Khammash (2012)) and the second applies a tracking method to converge to a target pdf. Both methods rely on on-line computation of the (change in) pdfs, which is a necessity due to the stochastic dynamics associated with Brownian motion. In parallel, output pdf shaping through control actions has been considered with relation to chaotic systems (Lasota and Mackey (2013)). One of the main approaches to obtain a solution there, is by solving the inverse Frobenius-Perron problem (IFPP)(Nie and Coca (2015); Góra and Boyarsky (1993)) to construct a discrete-time 1-D state transformation that has a given limiting pdf. Hence, the information obtained from the solution of the IFPP can be used to perturb the existing system dynamics in such a way that convergence to the desired pdf is achieved (Bollt (2000)). The contributions listed above provide interesting methods to deal with and evaluate stochastic states. Combining this notion with classical control for LTI systems then allows us deal with our deterministic evolution of the stochastic initial conditions.

In this paper, we propose two transient behaviour specifications that are suitable for deterministic systems with stochastic initial conditions. Subsequently, these specifications will be combined with a specification for the asymptotic behaviour of the system to form the basis of two control problems. Accordingly, the two control problems both specify an asymptotic converge criterion, while they

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require different performance in the transient, namely; the first control problem (CP1) will evaluate the cumulative distribution in an interval, while the second control problem (CP2) will evaluate the similarity of the distribution of the state with a desired distribution. Consequently, we obtain the following main contributions: (i) a proposed linear control design that can always solve CP1 for higher order systems and initial conditions that satisfy a normal distribution; (ii) the extension of (i) to CP2 for first order systems and initial conditions that satisfy a normal distribution; and (iii) the extension of (ii) to non-normal distributions, resulting in a non-linear controller.

The remainder of the paper is structured as follows. Section 2 introduces the considered system dynamics and the considered control problems CP1 and CP2. In Section 3, we evaluate simple controllers for CP1. Furthermore, this section contains a rigorous solution for CP1, that always exists for certain systems with initial conditions that satisfy a normal distribution. Subsequently, Section 4 will be used to present an extension of the results obtained Section 3 to CP2. In addition, we use this section to introduce an extension of the obtained solutions for CP2 to arbitrary distributions here. Lastly, Section 5 closes this work with conclusions.

2. PROBLEM DESCRIPTION AND PRELIMINARIES

Consider the following LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu,\ x(0) = x_0, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x(t) \in X \subset \mathbb{R}^n$ is the state, $u(t) \in U \subset \mathbb{R}^m$ is the control input, $y(t) \in Y \subset \mathbb{R}^q$ is the output and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{q\times n}$ are the system matrices. Here, we assume that the initial conditions x_0 are random variables with a given probability density function $f_{x_0} : X \mapsto \mathbb{R}_+$. Such a system description includes the standard deterministic model where f_{x_0} is given by a delta function, and it can capture uncertainties in many applications.

If we compare (1) with the standard systems formulation that is used in the literature of stochastic control systems, the latter case assumes typically that random variables influence the state equation as external disturbance while the former one introduces the random variable through the initial condition. Consequently, the control problem for stochastic systems is related to the design of controllers that can suppress the effect of such random disturbance signals with a prescribed level of attenuation. On the other hand, for systems such as (1), we assume for the moment that the disturbance signal is negligible and the uncertainty in the process comes mainly from the initial condition. In this case, we are interested in evaluating the evolution of f_{x_0} under the dynamics of (1) and for a given control law $u(t) = k(x(t), t)$. From now on, we will denote the time evolution of f_{x_0} by $f_{x_0,t}$.

In the following paragraphs, we will discuss two control problems where we are interested in designing a control law $u(t) = k(x(t), t)$ such that the density function at a prescribed time T, e.g. $f_{x_0,T}$, meets a given specification (which will be discussed shortly) and additionally, the standard asymptotic behaviour, where x converges to a desired state x^*, is still achieved. We remark here that the first control goal generalizes the standard transient behaviour requirement. Typically, we specify either a relative transient behaviour requirement or a strict (or conservative) transient specification. In the former case, we usually demand that the error signal (due to unknown initial state x_0) converges to a desired percentage level (which is commonly 1%, 5% or 10%) from its initial value at a prescribed transient time. On the other hand, for the latter case, a fixed funnel (of both the state and time) is defined and a control law is designed such that all state trajectories always remain in the funnel. For the design of such control law, we refer interested readers to Ilchmann and Schuster (2009).

Let us now describe two possible transient behaviour specifications on the evolution of the pdf $f_{x_0,t}$ which will be considered in this paper.

Transient behaviour specification 1. Given a compact subset $\Xi \subset X$ and given a transient time T, we define the *cumulative distribution* of x at time T on Ξ by

$$F_{\Xi,T} \equiv \int_{\Xi} f_{x_0,T}(\xi)\mathrm{d}\xi. \tag{2}$$

By the definition of pdf, $f_{x_0,t}$, $F_{\Xi,t} \in [0, 1]$ for all $t \geq 0$ and for all $\Xi \subset X$.

Transient behaviour specification 1 is highly relevant for applications that demand a prescribed confidence interval/domain at the transient time, when we have apriori knowledge on the distribution of the initial state. It is in line with many performance criteria that are encountered in practical applications.

Transient behaviour specification 2. Consider a desired pdf at transient time T by $f_d : X \mapsto \mathbb{R}_+$, we define the *Bhattacharyya distance* between $f_{x_0,T}$ and f_d by

$$d(f_d, f_{x_0,T}) \equiv -\ln\left(\int_X \sqrt{f_d(\xi) f_{x_0,T}(\xi)}\mathrm{d}\xi\right). \tag{3}$$

It can be checked that $d(f_d, f_{x_0,T}) \in [0, \infty)$, where it will be zero if $f_{x_0,T}$ and f_d are identical.

The Bhatacharrya distance is based on the Bhattacharyya coefficient (Buehler et al. (2016); Abou-Moustafa and Ferrie (2012)). Although transient behaviour specification 2 seems difficult to attain, it is suitable for applications that require a precision in the behaviour of the state's density function. One of the relevant examples is the shaping of a point spread function in an optical system where we can have apriori knowledge on the distribution of the point source and the optical train can be controlled such that the dynamic of the image point spread function can be shaped accurately. A photolithography system is an example of such application.

Remark 1. A disadvantage of the Bhattacharyya distance is that it is a semi-metric, since it does not satisfy the triangle inequality. For applications where this proves to be a problem, one could alternatively consider the Hellinger distance (Abou-Moustafa and Ferrie (2012)).

We are now ready to define two control problems based on the above transient behaviour specifications.

Control Problem 1. (CP1) For the system in (1) with transient time T, desired containment set Ξ, desired containment level $p^* \in (0,1)$ and desired steady state x^*, design a smooth control law $u(t) = k(x(t), t)$ s.t.

CP1a: $F_{\Xi,T} \geq p^*$
CP1b: $\lim_{t\to\infty} x(t) = x^*$. △

We will provide further explanations on the specifications **CP1a** and **CP1b** above. The specification **CP1a**, roughly speaking, implies that at transient time T, the probability that the state $x(T)$ is in the set Ξ, is at least p^*. The specification **CP1b** is the usual asymptotic convergence requirement of the state to the desired steady state x^*.

Control Problem 2. (CP2) For the system in (1) with transient time T, desired pdf f_d and maximum distance $\ell \in [0,\infty)$, design a smooth control law $u(t) = k(x(t), t)$ s.t.

CP2a: $d(f_d, f_{x_0,T}) \leq \ell$
CP2b: $\lim_{t\to\infty} x(t) = x^*$. △

Analogous to CP1, the specification **CP2a** implies that at the transient time T, the evolution of $f_{x_0,T}$ is nearly the desired pdf f_d, such that the Bhattacharyya distance is not greater than ℓ. Furthermore, **CP2b** guarantees the asymptotic convergence to x^*, as it is the case for **CP1b**.

3. CONTROL DESIGN WITH PRESCRIBED CONTAINMENT LEVEL

This section will be used to investigate controller designs to solve CP1. For simplicity of presentation, we will start to consider a first-order system where $n = m = q = 1$ and with $f_{x_0} = \mathcal{N}(\mu, \sigma)$, where $\mathcal{N}$ denotes the normal distribution with μ its mean value and σ its standard deviation. Using this simple first-order system as a first step will allow us to get some interesting insights about how to use classical control laws to solve CP1. As a second step, we extend the obtained results for first order systems to higher order systems.

3.1 CP1 for first-order LTI systems with a normally distributed initial condition

As a starting point, we will consider a standard linear control law, namely a state feedback, for solving CP1. From here, we will gradually expand our focus such that we can obtain a rigorous solution to CP1.

Since we consider a first-order LTI system (with $A = a \in \mathbb{R}$ and $B = b \in \mathbb{R}$), the application of a simple linear feedback

$$u = k(x - x^*) - \frac{a}{b}x^*, \tag{4}$$

with $k \in \mathbb{R}$ to (1) will lead us to the following simple expression of the closed-loop system

$$\dot{\tilde{x}} = (a + bk)\tilde{x},\ \tilde{x}(0) = x_0 - x^*, \tag{5}$$

where $\tilde{x} = x - x^*$ is the error state and the gain k can be chosen arbitrarily to ensure that $(a + bk) < 0$. Due to the fact that we assume a normal distribution for the initial state, we obtain $f_{\tilde{x}_0} = \mathcal{N}(\mu - x^*, \sigma)$. Since we are interested in a non-trivial solution of control problem 1, we assume that $\mu \neq 0$. For defining the first transient behavior specification of the closed-loop system (5), we take $\tilde{\Xi} = [x_{T,low}, x_{T,up}] - x^*$ where $x_{T,low}$ and $x_{T,up}$ are the lower and upper bound of the containment interval Ξ.

Since we are dealing with a simple first-order linear system, we can use the bounds of $\tilde{\Xi}$ and the explicit solution of (5) to construct the image of this containment interval at time $t = 0$, which we denote as $\tilde{\Xi}_0$. In this way, the value $F_{\tilde{\Xi},T}$ will be equivalent to cumulative distribution of $\tilde{x}_0$ on $\tilde{\Xi}_0$.

Based on the solution of (5), we have

$$\tilde{x}_{0,low} = e^{-(a+bk)T}\tilde{x}_{T,low} \tag{6}$$

and

$$\tilde{x}_{0,up} = e^{-(a+bk)T}\tilde{x}_{T,up}, \tag{7}$$

where, understandably, $\tilde{x}_{0,low}$ and $\tilde{x}_{0,up}$ are the lower and upper bound of $\tilde{\Xi}_0$.

Since $f_{\tilde{x}_0} = \mathcal{N}(\mu - x^*, \sigma)$, we can determine the maximum containment level p_{max} by solving

$$\begin{aligned} p_{\max} = \max_k \frac{1}{2}\Bigg[&\operatorname{erf}\left(\frac{e^{-(a+bk)T}\tilde{x}_{T,up} - \mu + x^*}{\sigma\sqrt{2}}\right) - \\ &\operatorname{erf}\left(\frac{e^{-(a+bk)T}\tilde{x}_{T,low} - \mu + x^*}{\sigma\sqrt{2}}\right)\Bigg], \end{aligned} \tag{8}$$

where erf is the error function. This quantity tells us that we will always have $F_{\tilde{\Xi},T} \leq p_{\max}$. This implies that if $p_{\max} < 1$, we cannot solve CP1 for arbitrary containment level $p^* \in (0,1)$.

In the following numerical example, we will demonstrate a case where a simple linear feedback cannot solve CP1 for an arbitrary containment level.

Example 1. Consider a robotic swarm case where we assume that each agent satisfies Newton's second law of motion with unitary mass for a 1-D case, and we can directly control the velocity of the agents as follows

$$\dot{x} = u, \qquad x(0) = x_0, \tag{9}$$

where we assume that $f_{x_0} = \mathcal{N}(10, 1)$. For simplicity, we neglect the collision among agents and we consider the desired containment set $\Xi = [4, 5]$ with $T = 5$.

Firstly, let us consider a non-zero desired equilibrium point of $x^* = 4$. Using the linear feedback controller as given before, we can obtain the gain $k < 0$ for any desired containment level $p^* \in (0,1)$. For instance, by taking $k = -3.6776$, we get p^* very close to 1. Since $k < 0$, the closed-loop system is stable which implies that $x(t)$ converges to x^* as $t \to \infty$. Hence we achieve both **CP1a** and **CP1b**.

On the other hand, if we change the desired steady-state to $x^* = 0$ then the aforementioned feedback control will no longer solve CP1 for arbitrary p^*. The main reason for this is that we can no longer design k such that **CP1a** is met for some desired containment level p^*. Indeed, solving (8) results in $p_{\max} = 0.7359 < 1$ which occurs for $k = -0.1617$. Hence, we can no longer find a feasible solution that satisfies both **CP1a** and **CP1b**. △

In Example 1, we have shown that the previous simple linear feedback control law only allows us to solve CP1 for specific cases. Particularly, achieving a desired containment level p^* close to 1 may not be possible at all. This problem can be exacerbated later when we are interested to solve control problem 2 where we want to achieve a target pdf during the transient time.

In the following discussion, we will present a simple solution to the above issue, by introducing a feedforward signal. Generally speaking, the basic idea is to define first an admissible tracking reference signal x_r such that: (i). $\lim_{t\to\infty} x_r(t) = x^*$; (ii). $x_r(0) \in X$, $x_r(T) \in \Xi$; and (iii). there exists a feedback gain k such that the closed-loop system is contracting (towards x_r) with a desired convergence speed based on the given transient time T.

More precisely, using such a tracking reference signal x_r, let us consider now the well-known proportional feedback control law

$$u(t) = k(x(t) - x_r(t)) + \dot{x}_r. \tag{10}$$

One of the plausible choices of x_r is an exponential function satisfying $x_r(0) = \mu$, $x_r(T) = 0.5(x_{T,low} + x_{T,up})$ and $\lim_{t\to\infty} x_r(t) = x^*$.

Example 2. Recall the robotic swarm dynamics from Example 1, and suppose now that we want to have a desired containment level of $p^* = 0.95$ with the desired steady-state $x^* = 0$, with $T = 5$ and with desired containment set $\Xi = [4,5]$. As shown before in Example 1, it is not possible to attain $p^* = 0.95$ using the control law in (4) for any gain k.

As discussed briefly before, we consider now the new control law as in (10). The desired tracking reference signal, x_r has to satisfy (i). $x_r(0) = \mu$; (ii). $x_r(T) = 0.5(x_{T,low} + x_{T,up})$; and (iii). $x_r(t) = 0$ as $t \to \infty$. Using such x_r, the gain k can be chosen such that we achieve the desired contraction at time T.

Let us consider the following tracking reference signal x_r that satisfies the above requirements:

$$x_r(t) = \mu e^{\frac{\ln\left(\frac{x_r(T)}{x_r(0)}\right)}{T} t} = 10e^{-0.1597t}. \tag{11}$$

We can calculate the gain k by determining the correct exponential rate so that the interval of initial error state $\tilde{\Xi}_0 = [-2\sigma, 2\sigma]$ will contract to the desired interval of $\tilde{\Xi}_T = [x_{T,low} - x_r(T), x_{T,up} - x_r(T)] = 0.5[x_{T,low} - x_{T,up}, x_{T,up} - x_{T,low}]$ at time T. Note that the cumulative distribution on $\tilde{\Xi}_0$ is approximately 0.955. More precisely, the gain k can be given by

$$k = \frac{\ln\left(\frac{0.5(\tilde{x}_{T,up} - \tilde{x}_{T,low})}{\mathrm{erf}^{-1}(p^*)\sqrt{2}\sigma}\right)}{T} = -0.2732, \tag{12}$$

where erf^{-1} is the inverse error function.

Using the control law in (10) where k and x_r are as given above, we solve the CP1, and particularly, we achieve $F_{\tilde{\Xi},T} \approx p^*$.

Using a numerical simulations with 10000 samples of initial conditions following the given normal distribution, we compare the distribution of the state at the transient time T (i.e., $f_{x_0,T}$ with the desired normal distribution f_d, which has a mean $x_r(T) = 0.5(x_{T,low} + x_{T,up}) = 4.5$ and a standard deviation $\sigma \frac{0.5(x_{T,up} - x_{T,low})}{\mathrm{erf}^{-1}(p^*)\sqrt{2}\sigma} = 0.2551$; this is shown in Fig. 1. Based on this simulation, the calculated Bhattacharyya distance is $d(f_d, f_{x_0,T}) \approx 0$, which means that we are very close to the desired distribution at time T. The time evolution of the mean and standard deviation corresponding to $f_{x_0,t}$ are shown in Fig. 2. △

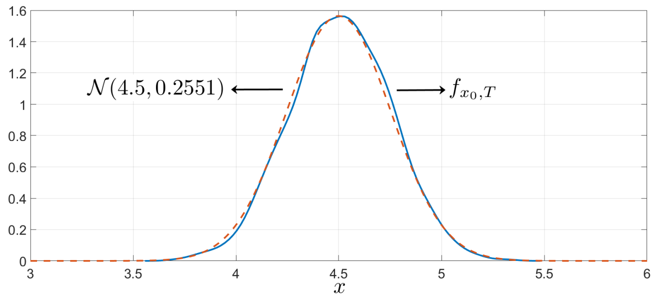

Fig. 1. This figure corresponds to example 2 and shows a comparison between: i) the obtained result for a feed-forward controlled system with a normal initial pdf f_{x_0}, converging to the closed interval $\Xi = [4,5]$; and ii) a normal PDF with mean 4.5 and standard deviation 0.2551. The two distributions have a Bhattacharyya distance of $d(f_d, f_{x_0,T}) \approx 0$

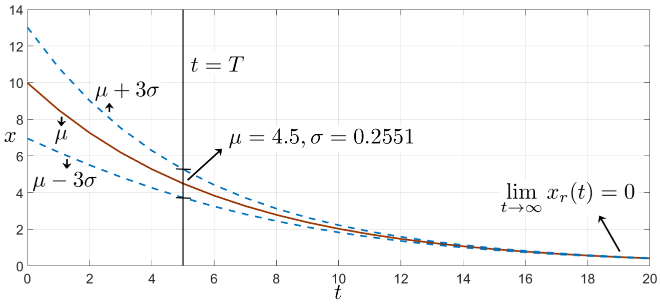

Fig. 2. This figure shows the time evolution of the mean μ and the standard deviation σ corresponding to $f_{x_0,t}$ in Example 2. The graph shows that an exponential convergence to $x^* = 0$ is achieved, while satisfying $F_{\Xi,T} \geq p^*$.

In example 2, we have shown that by introducing a proper tracking reference signal x_r, the control law (10) can solve CP1 with a proper design of gain k, which could not be solved before in Example 1, where control law (4) is considered.

3.2 CP1 for higher-order LTI systems with normally distributed initial conditions

Based on Example 2, we will now extend the obtained result to the general multivariate system description as in (1). Here, the initial condition is a random variable following a multivariate normal distribution. This is formalized in the following proposition.

Proposition 1. Consider the system as in (1) where $f_{x_0} = \mathcal{N}(\mu, \Sigma)$, the mean vector $\mu \in \mathbb{R}^n$ and the co-variance matrix $\Sigma \in \mathbb{R}^{n\times n}$. Assume that the pair (A, B) is controllable. Then, CP1 is solvable for any T, μ, Σ, p^*, x^* with Ξ being compact and connected.

Proof. The proof of the proposition follows a similar line as the one for the first order system in the example above. Consider the control law

$$u(t) = K(x(t) - x_r(t)) - u^*(t), \tag{13}$$

where x_r and u^* are the tracking reference signal and additional feedforward input signal to be designed.

Since Σ is compact and connected, we can define a closed ball centered in ϵ with radius δ (which we denote by $\mathbb{B}_\delta(\epsilon)$) that is contained in Σ, i.e., $\mathbb{B}_\delta(\epsilon) \subset \Sigma$. Define x_r and u^* with the following properties: (i). $\lim_{t\to\infty} x_r(t) = x^*$; (ii). $x_r(0) = \mu$; (iii). $x_r(T) = \epsilon$; and $\dot{x}_r(t) = Ax_r(t) + Bu^*(t)$. Note that since the pair (A, B) is controllable, we can always find a control signal u^* that can bring the state from μ at time 0 to ϵ at time T, and subsequently, to x^* at infinity. Using such u^*, the tracking reference signal x_r is then given by the solution z of

$$\dot{z} = Az + Bu^*, \qquad z(0) = \mu.$$

Define now $\zeta = x - x_r$ as the error signal between the state and the tracking reference signal. Note that with such coordinate transformation, if $\zeta(T) \in \mathbb{B}_\delta(0)$ then, since $x_r(T) = \epsilon$, it implies that $x(T) \in \mathbb{B}_\delta(\epsilon)$, which is contained in Σ. Also, since $f_{x_0} = \mathcal{N}(\mu, \sigma)$ and $x_r(0) = \mu$, it follows immediately that $f_{\zeta_0} = \mathcal{N}(0, \sigma)$.

Hence, the dynamics of the error signal where we have applied the control law (13) is simply given by

$$\dot{\zeta} = (A + BK)\zeta, \qquad \zeta(0) = \zeta_0. \tag{14}$$

Let us now define a ball of initial condition $\mathbb{B}_\gamma(0)$, where γ is chosen such that $\gamma > \delta$ and

$$\int_{\mathbb{B}_\gamma(0)} f_{\zeta_0}(\xi)\mathrm{d}\xi > p^*. \tag{15}$$

We furthermore take a contraction exponential rate constant $\lambda = -\frac{1}{T}\ln(\delta/\gamma)$. In the following, we will design K so that $\mathbb{B}_\gamma(0)$ under the closed-loop dynamics (14) will be contracted with an exponential rate of λ to $\mathbb{B}_\delta(0)$ at time T.

From the pair (A, B) being controllable, it follows that we can design K such that the eigenvalues of $A + BK$ whose real parts are less than $-\lambda$ (for example, by the well-known pole-placement method). This implies that

$$\|\zeta(t)\| \le e^{-\lambda t}\|\zeta(0)\|$$

holds for all initial condition $\zeta(0)$. By our choice of λ as given before,

$$\|\zeta(T)\| \le \frac{\delta}{\gamma}\|\zeta(0)\|.$$

Hence,

$$\zeta(0) \in \mathbb{B}_\gamma(0) \Rightarrow \zeta(T) \in \mathbb{B}_\delta(0) \Rightarrow x(T) \in \mathbb{B}_\delta(\epsilon) \subset \Xi.$$

Since (15) holds, the above implications mean also that

$$F_{\Xi,T} \ge \int_{\mathbb{B}_\gamma(0)} f_{\zeta_0}(\xi)\mathrm{d}\xi > p^*.$$

In other words, **CP1a** is satisfied. Additionally, the following asymptotic property also holds:

$$\lim_{t\to\infty} \zeta(0) = 0 \Rightarrow \lim_{t\to\infty} x(t) = \lim_{t\to\infty} x_r(t) = x^*.$$

This means that **CP1b** holds. This concludes the proof.

□

Roughly speaking, the main idea of the control design method as given in the proof of Proposition 1 is based on the designing a proper contraction of a ball of initial conditions, whose cumulative distribution is larger than p^*, to a smaller ball inside the containment set Ξ with the help of a reference signal x_r. Based on this design principle, we can also extend the method to nonlinear systems, where we can apply recent results on contraction principle for nonlinear systems (for example, in (Andrieu et al. (2016))).

4. CONTROL DESIGN WITH PRESCRIBED TARGET DISTRIBUTION

In this section, we will consider now control design for solving CP2 where we want to reach a target distribution at the transient time with a prescribed distance (which is defined using the Bhattacharyya distance). Without loss of generality, let us consider the system (1) where $n = 1$ (i.e., first-order system).

4.1 CP2 for first-order LTI systems with a normally distributed initial condition and normal desired distribution

The following proposition follows directly from Proposition 1 and it will be useful later when we consider a broader class of distribution functions for the initial condition, as well as, for the target distribution.

Proposition 2. Assume that the hypothesis of Proposition 1 holds where $n = 1$, $f_{x_0} = \mathcal{N}(\mu, \sigma)$ with $\sigma > 0$ and $\mu \in \mathbb{R}$. Suppose that the target distribution at time $T > 0$ is given by $f_d = \mathcal{N}(\mu_d, \sigma_d)$ with $0 < \sigma_d < \sigma$. Then CP2 is solvable for any given similarity level $\ell \in [0, \infty)$.

Proof. The proof follows a similar line to that of Proposition 1. Let us consider x_r as given in the proof of Proposition 1 and since we consider first-order system, we denote $A = a \in \mathbb{R}$, $B = b \in \mathbb{R}$ and $K = k \in \mathbb{R}$. By denoting $\zeta = x - x_r$, following the same derivation as in the proof of Proposition 1, we have

$$\dot{\zeta} = (a + bk)\zeta, \qquad \zeta(0) = \zeta_0. \tag{16}$$

For this error system, the specification for transient behavior becomes $f_d^\zeta = \mathcal{N}(0, \sigma_d)$ for the target distribution at time T and $f_{\zeta_0} = \mathcal{N}(0, \sigma)$ for the initial distribution. Using this error system, CP2 can be solved by designing k such that the initial standard deviation σ contracts close to σ_d at the transient time T (determined by the maximum distance ℓ).

If we denote the evolution of standard deviation by s, it is straightforward to check that σ satisfies

$$\dot{s} = (a + bk)s, \qquad s(0) = \sigma. \tag{17}$$

Thus, we need to design k such that $s(T)$ is close to σ_d so that the condition **CP2a** holds. In this case, we do not necessarily have to determine the exact value of k so that $s(T) = \sigma_d$.

Using the analytical expression of the Bhattacharyya distance for two normal distributions (Coleman and Andrews (1979)), the distance between $f_d^\zeta = \mathcal{N}(0, \sigma_d)$ and $f_{\zeta_0,T} = \mathcal{N}(0, s(T))$ is given by

$$d(f_d^\zeta, f_{\zeta_0,T}) = \frac{1}{4}\ln\left(\frac{1}{4}\left(\frac{\sigma_d^2}{s^2(T)} + \frac{s^2(T)}{\sigma_d^2} + 2\right)\right). \tag{18}$$

When $s(T) = \sigma_d$ (i.e., we reach the target distribution precisely) then $d(f_d^\zeta, f_{\zeta_0,T}) = 0$, otherwise it is strictly greater than 0.

Thus, given a maximum distance ℓ, we can calculate $\epsilon > 0$ s.t. the following implication

$$\epsilon \geq s(T) \geq \sigma_d \ \Rightarrow d(f_d^\zeta, f_{\zeta,T}) \leq \ell$$

holds.

Finally, by using ϵ, we can determine the range of the gain k such that the evolution of s as in (17) will satisfy $\epsilon \geq s(T) \geq \sigma_d$. □

4.2 CP2 for first-order LTI systems with a uniformly distributed initial condition and normal desired distribution

Let us now consider the case that we have arbitrary distributions in our transient behavior specifications f_{x_0} and f_d which are not necessarily normal distribution. As before, we consider first-order systems $\dot{x} = ax + bu$. Using the result in Proposition 2, an intuitive solution to this problem is to find an appropriate common coordinate transformation such that both specifications f_{x_0} and f_d become normal distribution and we can use tools from contraction theory (such as, the results presented in Andrieu et al. (2016)) or standard state linearization method to get the appropriate contraction of the initial normal distribution to the target normal distribution.

More precisely, suppose that there exists a diffeomorphic map $\Psi : X \to Z$ such that if we denote $z = \Psi(x)$, then the transformed distributions

$$f_{x_0} \overset{\Psi}{\mapsto} f_{z_0}$$
$$f_d \overset{\Psi}{\mapsto} f_d^z$$

are both normal distributions.

Then, using the state transformation $z = \Psi(x)$, the transformed system is given by

$$\dot{z} = \Phi(z)(a\Psi^{-1}(z) + bu), \qquad z(0) = z_0 = \Psi(x_0), \tag{19}$$

where $\Phi(z) = \frac{\partial \Psi}{\partial x}|_{x=\Psi^{-1}(z)}$. Hence, choosing

$$u = b^{-1}a\Psi^{-1}(z) + \Phi^{-1}(z)\tilde{u} \tag{20}$$

gives us the linearized system

$$\dot{z} = \tilde{u}.$$

Using this linearized system, we can implement the control design as given in the proof of Proposition 2 for solving CP2.

In the following example, we will show the numerical implementation of such controller.

Example 3. Let us consider again the exemplary system of swarming robots with unit mass, similar to Example 1. This time, we are interested in convergence of a standard uniform pdf defined on $X = [0, 1]$, e.g. $f_{x_0} : [0, 1] \mapsto 1$, to a truncated normal pdf $f_d : [0, 1] \mapsto \mathbb{R}_+$ with a mean value $\mu = 0.7$ and a standard deviation $\sigma = 0.02$. We furthermore require a Bhattacharyya distance of at most 0.05, e.g. $\ell = 0.05$, at time $T = 10$, and $x^* = 0.8$.

We then apply a coordinate transformation $z = \Psi(x)$ that transforms both distributions to (an approximate) normal distribution, belonging to a system driven by the dynamic equation

$$\dot{z} = \tilde{u} \tag{21}$$

In this case, such a transformation and its inverse can be

$$z = \Psi(x) = 0.1\,\mathrm{erf}^{-1}(2x - 1)\sqrt{2} + 0.5, \tag{22}$$

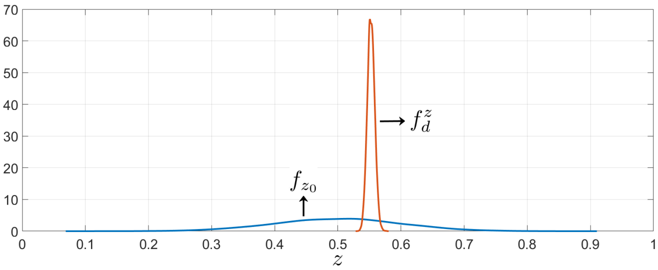

Fig. 3. This figure shows the pdfs of the transformed initial and target distribution used in Example 2. We use this figure to show that the transformed distributions strongly resemble normal distributions and that both a shift of the mean and a reduction of the standard deviation is required for convergence.

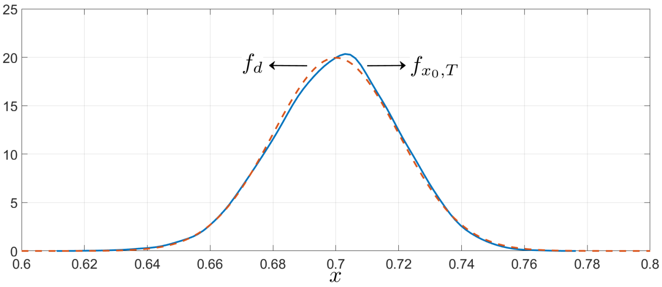

Fig. 4. This figure shows, for comparison, the desired pdf f_d and the achieved pdf $f_{x_0,T}$ at time T, as obtained from Example 2. We use this figure to show that the two pdfs strongly overlap. These two pdfs have a Bhattacharyya distance of $d(f_{x_0,T}, f_d) \approx 0$.

$$x = \Psi^{-1}(z) = 0.5\left[1 + \mathrm{erf}\left(\frac{z - 0.5}{0.1\sqrt{2}}\right)\right], \tag{23}$$

from which we can construct the input to the original systems as

$$u = \frac{\partial \Psi}{\partial x}|_{x=\Psi^{-1}(z)}\tilde{u} \tag{24}$$

From the transformation, we acquire the two (approximate) normal pdfs f_{z_0} and f_d^z. These distributions are shown in fig. 3 and have corresponding means $\mu_0 = 0.5$ and $\mu_d = 0.55$, and standard deviations $\sigma = 0.1$ and $\sigma_d = 0.006$. We apply the control law $\tilde{u}(t) = ke_z(t) + \dot{z}_r$, $e_z(t) = \tilde{y}(t) - \tilde{x}_r(t)$ and $k = \frac{\ln\left(\frac{\sigma_d}{\sigma}\right)}{T} = -0.2852$. The reference signal z_r should satisfy $z_r(0) = \mu_0$, $\tilde{x}_r(T) = \mu_d$ and $\lim_{t\to\infty} z_r(t) = \Psi(x^*) = 0.5842$. Hence, we can use $z_r(t) = 0.5842 - 0.0842e^{-0.0902t}$. The input $u(t)$ of the original system is computed through (24). The result at time T in comparison with f_d can be seen in Fig. 4. The evolution of the mean and standard deviation of $f_{z_0,t}$ over time is shown in fig. 5. We have $d(f_{x_0,T}, f_d) \approx 0$ and therefore satisfy $d \leq \ell$. Furthermore, as $\lim_{t\to\infty} z_r(t) = 0.5842$, we have that $\lim_{t\to\infty} x(t) = 0.8$. We have therefore solved CP2. □

As we have shown in Example 3, using the transformation enables us to solve CP2 for convergence of an initial uniform distribution to a desired normal distribution for a first-order system. This can potentially work for all

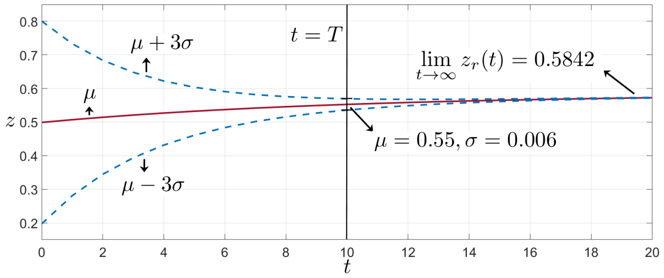

Fig. 5. This figure shows the time evolution of the mean μ and the standard deviation σ corresponding to $f_{z_0,t}$ in Example 3.

distributions, as long as we are able to find a proper coordinate transformation.

5. CONCLUSIONS

We have investigated controller design options to achieve desired transient and asymptotic behaviour for deterministic systems with stochastic initial conditions. This work has been structured through the formulations of two control problems, which both require a specific asymptotic performance and where, at a specific transient time, the first (CP1) requires a minimal cumulative distribution on a subset, while the second (CP2) requires a maximal distance to a desired distribution. We then obtained the following results: (i) A linear controller that can always solve CP1 for high-order systems with normal initial distributions; (ii) An extension of (i) such that this controller can always CP2 for first-order systems and where the desired distribution is also a normal distribution; (iii) An extension of (ii) to achieve the convergence from a non-normal initial distribution (namely, a uniform distribution) to a desired normal distribution through a coordinate transformation that causes the controller to become nonlinear.

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